Acemoglu’s Book solutions
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Macroeconomics theory

This is a supplementary material for the Peters and Simsek’s solution book. My intention is provide some alternative approaches in order to solve the Acemoglu’s questions or just fill out the missing lines you can find in the solution book. Most of the time the algebra for book solutions demands room or space for making the answer elegant or simple. Here we do not take care about room or space.

2.1 Taking Euler theorem and supposing competitive labor and capital markets we have that \( Y = wL + rK \). Re-arranging former expression and expressing wage in terms of production and capital we have that \( w = f(k) - rf\left(\frac{K}{K}\right) \). Then by using the competitive markets assumption \( w = f(k) - f'(k)k \). We want to know what are conditions for imposing to have \( w > 0 \); in this sense we have that \( f(k) > f_0(k)k > 0 \) or \( f(k) > f_0(k)k > 1 \). then we claim that \( f(k) > f_0(k)k > 1 \) because of assumption 1, specifically the assumption respect to constant returns to scale.

2.2 By assumption 1 we have that \( F \) exhibits returns to scale, positive diminishing-marginal product for each input and differentiability. Accordingly, a real valued function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is concave if

\[
\lambda^{-1} f(\lambda x + (1 - \lambda) y) \geq f(x) + (1 - \lambda) \frac{1 - \lambda}{\lambda} f(y)
\]

then \( y = Ax \), where \( A \in \mathbb{R}^n \times \mathbb{R}_+ \). A particular matrix \( A = aI \) where \( a \in \mathbb{R}_+ \). We have an additional assumption: Goods markets are competitive. Let’s set up the profit function:

\[
\pi(K, L) = P \cdot F(K, L) - rK - wL
\]
in matricial terms we have $\pi = F(x) - P_z \cdot x$. So, $x = \{K, L\} \in \mathbb{R}_+^2 = \text{arg max} \pi$. Then, we setup Karush-Kuhn-Tucker (KKT) conditions

$$\mathcal{L} = P \cdot F(K, L) - rK - wL + \lambda K + \gamma L$$

where $\lambda, \gamma$ are the KKT lagrangian multipliers\(^1\). The first order conditions (f.o.c) are

$$\frac{\partial \mathcal{L}}{\partial K} = P \cdot F_K - r + \lambda = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial K} = P \cdot F_L - w + \gamma = 0 \quad (3)$$

The slackness conditions $\lambda K = 0$ and $\gamma L = 0$. Let’s look through some production function which exhibits returns to scale. For example, $\tilde{F} = K^\alpha L^{1-\alpha}$ in that case we are going to have combining the f.o.c in expressions (2) and (3). In that case we have

$$K = \frac{w - \gamma}{r - \lambda} \left( \frac{\alpha}{1 - \alpha} \right)$$

Then $K/L = \kappa$, any $K/L$ for some $\kappa > 0$ is a solution, being $K, L > 0$ which implies $\gamma, \lambda = 0$. But what if we choose another pair from the following set $A = \{\{K, L\} \in \mathbb{R}_+^2 \mid \frac{K}{L} = \left( \frac{w - \gamma}{r - \lambda} \right) \left( \frac{\alpha}{1 - \alpha} \right) > 0\}$ then the profit function becomes

$$\pi_m = (1 + m)^\alpha F(K, L) - (1 + m) rK - wL$$

Then

$$\frac{\partial \pi_m}{\partial m} = \alpha (1 + m)^{\alpha-1} F(K, L) - rK = 0$$

$$= \alpha (1 + m)^{\alpha-1} \frac{F(K, L)}{L} - \frac{rK}{L}$$

Then including the f.o.c

$$\alpha (1 + m)^{\alpha-1} \frac{F(K, L)}{L} = \frac{rK}{L}$$

$$\alpha (1 + m)^{\alpha-1} \frac{F(K, L)}{L} = r \left( \frac{w}{r} \right) \left( \frac{\alpha}{1 - \alpha} \right)$$

$$= \frac{wL}{F(K, L)} \left( \frac{1}{1 - \alpha} \right)$$

$$= (1 + m)^{\alpha-1} \left( 1 - \alpha \right) \left( \frac{1}{1 - \alpha} \right)$$

the solution is $m = 0$. So, the pair $K, L$ fulfilling $K/L = \kappa > 0$ is a solution.

Now, let’s try with other production function which exhibits returns to scale. Let’s say $F(K, L) = aK + bL$, with $r > a$ and $w > b$ So the lagrangian is

$$\mathcal{L} = aK - bL - rK - wL + \lambda K + \gamma L$$

\(^1\)In fact they have received different names like dual variables, dual prices or shadow prices.
the first order conditions are:

\[
\frac{\partial L}{\partial K} = a - r + \lambda = 0
\]
\[
\frac{\partial L}{\partial K} = b - w + \gamma = 0
\]

or

\[
a - r = -\lambda
\]
\[
b - w = -\gamma
\]

We satisfy above conditions if \( K = L = 0 \Rightarrow \lambda, \gamma > 0 \). If \( a - r > 0 \) and \( b - w > 0 \) then conditions in (4) and (5) are not satisfied providing that lagrangians must be positives and according to KKT conditions there is no a solution.

2.11 We have the following assumptions: There is no population growth \( \frac{L}{L_0} = 0, \alpha + \beta < 1 \) and \( s \) is exogenously given

(a) We need to find a capital-labor ratio and output level in the steady state.

\[
F (K, L, Z) \equiv \frac{Y}{L} = K^\alpha L^\beta Z^{1-\alpha-\beta}
\]

In per capita terms \( y \equiv \frac{Y}{L} = \frac{K^\alpha L^\beta Z^{1-\alpha-\beta}}{L} \equiv \frac{K^\alpha Z^{1-\alpha-\beta}}{L} \equiv k^{1-\alpha-\beta} \) where \( z \equiv \frac{Z}{L} \). Then, we have the dynamic equation of capital or law of capital accumulation

\[
\dot{K} = -\delta K + I
\]

In per capita terms we have that \( \dot{k} = -\delta k + i \). Combining expressions (6) and (7) we have the following non-linear differential equation

\[
\dot{k} = -\delta k + sk^{1-\alpha-\beta}
\]

where we have used the following expression \( Y \equiv I = sF (K, L, Z) \). We do a change in variable in order to solve the differential equation, it is \( x = k^{1-\alpha} \). Thus, \( \dot{x} = (1 - \alpha)k^{-\alpha}k \). expression (8) can be re-expressed as:

\[
(1 - \alpha)k^{-\alpha}\dot{k} = -\delta (1 - \alpha)k^{-\alpha}k + s(1 - \alpha)k^{-\alpha}z^{1-\alpha-\beta}
\]
\[
\dot{x} = -\delta (1 - \alpha)x + s(1 - \alpha)z^{1-\alpha-\beta}
\]

above expression is a non-homogenous linear differential equation with non-time-varying coefficients. We are going to use the method of undetermined coefficients\(^2\) in order to solve this differential equation\(^3\). The guess\(^4\) is

\[
x = H_0 + H_1 e^{-H_2 t}
\]

\( x(0) \) given

We do the following calculations in order to get the parameters \( H_0, H_1 \) and \( H_2 \).

\[
\dot{x} = -H_2 H_2 e^{-H_2 t}
\]
\[
= -H_2 (x - H_0)
\]
\[
= -H_2 x + H_2 H_0
\]

\(^2\)“Guess and verify” method


\(^4\)In fact, that is not a guess-and-try method; it is a ad-hoc solution for that type of differential equation.
So, \( H_2 = \delta (1 - \alpha) \), \( H_0 = \frac{s z^{1-\alpha-\beta} \delta}{s} \). Then, we have that expression (1) becomes \( x = \frac{sx^{1-\alpha-\beta}}{s} + H_1 e^{-\delta(1-\alpha)t} \). The constant \( H_1 \) is determined by starting conditions; we know that \( x(0) = sz^{1-\alpha} + H_1 \). Thus we have the solution for the differential equation in expression (9)

\[
x = \frac{sz^{1-\alpha}}{s} + \left[x(0) - \frac{sz^{1-\alpha}}{s} \right] e^{-\delta(1-\alpha)t}
\]

Having the differential equation for \( k \)

\[
k = \left( \frac{sz^{1-\alpha}}{s} \right) + \left[ k(0) e^{-\gamma} - \frac{sz^{1-\alpha}}{s} \right] e^{-\delta(1-\alpha)t} \right)^{\frac{1}{\gamma}}
\]

The steady state of \( k \) is unique and determined by \( s, z \) and \( \delta \)

\[
\lim_{t \to \infty} k \equiv k^{ss} = \left( \frac{sz^{1-\alpha}}{s} \right) \left( \frac{1}{\gamma} \right)
\]

the per capita output level in steady state is

\[
\lim_{t \to \infty} y = \left( \frac{sz^{1-\alpha}}{s} \right) \frac{1}{\gamma} z^{1-\alpha-\beta}
\]

the output level is \( Y^{ss} \equiv \gamma^{ss} L = \left( \frac{s}{s} \right) \frac{1}{\gamma^s} z^{1-\alpha-\beta} L \). Now, we are going to check if that steady state is stable.

\[
\frac{\partial x}{\partial t} \bigg|_{x(0)} = \delta (1 - \alpha) \left[ x(0) - \frac{sz^{1-\alpha-\beta}}{s} \right] e^{-\delta(1-\alpha)t} > 0
\]

\[
\frac{\partial^2 x}{\partial t^2} \bigg|_{x(0)} = \delta^2 (1 - \alpha)^2 \left[ x(0) - \frac{sz^{1-\alpha-\beta}}{s} \right] e^{-\delta(1-\alpha)t} < 0
\]

So, the starting value you have chosen determines the shape of the path whose shape converge to the steady state. Thus, if \( x(0) - x^{ss} > 0 \) then \( \frac{\partial x}{\partial t} \bigg|_{x(0)} < 0 \) and \( \frac{\partial^2 x}{\partial t^2} \bigg|_{x(0)} > 0 \), this gives us information that the path is a strict convex function respect to \( t \). But if \( x(0) - x^{ss} < 0 \) then \( \frac{\partial x}{\partial t} \bigg|_{x(0)} > 0 \) and \( \frac{\partial^2 x}{\partial t^2} \bigg|_{x(0)} < 0 \), this gives us information that the path is a strict concave function respect to \( t \). Thus, the first and second derivative give us information about the stability of the system: does not matter where the initial condition is always we can guarantee a convergence to the steady state. Acemoglu’s solution books evaluates stability by checking the behavior of the growth rate of \( x \) by using the differential equation in (9); we can express that equation in the following terms

\[
\frac{x}{x} = -\delta (1 - \alpha) + \frac{sz^{1-\alpha-\beta}}{s}
\]

then we can say that if \( x(0) - x^{ss} > 0 \) then \( \frac{x}{x} < 0 \) because \( -\delta (1 - \alpha) > \frac{sz^{1-\alpha-\beta}}{s} \) that rule on the growth rate of \( x \) shows convergence. Also, if \( x(0) - x^{ss} < 0 \) then \( \frac{x}{x} > 0 \) because \( -\delta (1 - \alpha) < \frac{sz^{1-\alpha-\beta}}{s} \) that rule on the growth rate of \( x \) shows convergence
The general shape of this kind of differential equation is

\[ \frac{d}{dt} \left( \frac{Y}{L} \right) = K^\alpha Z^{1-\alpha-\beta} \]

where \( z \equiv \frac{Z}{L} \). Also we have that law of capital motion in per-capita terms is defined as

\[ \dot{K} \]

we conduct the same linearization procedure as we did in item \( \text{(a)} \) and after that try to deal with the time variant coefficient. the fact that

then we replace \( i \) in terms of \( z \) and \( k \). We re-express expression (11) as

\[ \dot{k} = - (\delta + n) k + s (1-\alpha) z^{1-\alpha-\beta} k^\alpha \]

what we have here is a non-linear differential equation with time variant coefficients\(^5\). You must remember that \( z \) depends on time; \( z \equiv \frac{Z}{L} = ZL_0 e^{-\alpha t} \), thus \( z = o(t) \). First at all we need to linearize expression (12) and after that try to deal with the time variant coefficient.

We conduct the same linearization procedure as we did in item \( \text{(a)} \) and we express explicitly the fact that \( z \) depends on growth rate \( n \).

\[ (1-\alpha) k^{-\alpha} \dot{k} = - (\delta + n) (1-\alpha) k^{-\alpha} k + s (1-\alpha) k^{-\alpha} k^\alpha z^{1-\alpha-\beta} \]

\[ x = - (\delta + n) (1-\alpha) x + s (1-\alpha) z^{1-\alpha-\beta} \]

\[ = - (\delta + n) (1-\alpha) x + s (1-\alpha) z(0)^{1-\alpha-\beta} e^{-n(1-\alpha-\beta)t} \]

\[ = - (\delta + n) (1-\alpha) x + \gamma e^{-n(1-\alpha-\beta)t} \]

Where \( \gamma = s (1-\alpha) z(0)^{1-\alpha-\beta} \). Then, in order to solve the differential equation we multiply the expression by \( e^{(\delta+n)(1-\alpha)t} \)

\[ x e^{(\delta+n)(1-\alpha)t} = - (\delta + n) (1-\alpha) x e^{(\delta+n)(1-\alpha)t} + \gamma e^{-n(1-\alpha-\beta)t} e^{(\delta+n)(1-\alpha)t} \]

we take notice that left side of expression is the total differential of expression \( x e^{(\delta+n)(1-\alpha)t} \), thus we have that

\[ \int_{t_0}^t \frac{\partial}{\partial s} \left[ x e^{(\delta+n)(1-\alpha)s} + c_0 \right] ds = \gamma \int_{t_0}^t e^{n\beta + \delta (1-\alpha)s} ds \]

Then taking integrals and solving it

\[ x e^{(\delta+n)(1-\alpha)t} + c_0 = \frac{\gamma}{n\beta + \delta (1-\alpha)} e^{n\beta + \delta (1-\alpha)s} \bigg|_{s=t}^{s=t_0} + c_1 \]

\[ x e^{(\delta+n)(1-\alpha)t} = \frac{\gamma}{n\beta + \delta (1-\alpha)} e^{n\beta + \delta (1-\alpha)s} \bigg|_{s=t_0}^{s=t} + c \]

\(^5\) The general shape of this kind of differential equation is \( \dot{x} = b(t) x + c(t) \).
where \( c_0, c_1 \) and \( c \) are constant of integration, being \( c \equiv c_1 - c_0 \). Re-arranging terms

\[
x = \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} e^{\left[ n \beta + \delta \left( 1 - \alpha \right) \right] s} \bigg|_{s=t}^{s=t_0} + ce^{-\left( \delta + n \right) \left( 1 - \alpha \right) t}
\]

(13)

Re-arranging terms and having \( t_0 = 0 \)

\[
x = \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} e^{-n \left( 1 - \alpha - \beta \right) t} - \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} e^{-\left( \delta + n \right) \left( 1 - \alpha \right) t} + ce^{-\left( \delta + n \right) \left( 1 - \alpha \right) t}
\]

gathering expressions and simplifying them, in particular \( n \beta + \delta \left( 1 - \alpha \right) - \left( \delta + n \right) \left( 1 - \alpha \right) \equiv -n \left( 1 - \alpha - \beta \right) \)

\[
x = \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} e^{-n \left( 1 - \alpha - \beta \right) t} + \left( x \left( 0 \right) - \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} \right) e^{-\left( \delta + n \right) \left( 1 - \alpha \right) t}
\]

constant \( c \) needs to be defined; given some starting value we have that \( x \left( 0 \right) = c \) when \( t_0 = 0 \). Then, we have that

\[
x = \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} e^{-n \left( 1 - \alpha - \beta \right) t} + \left[ x \left( 0 \right) - \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} \right] e^{-\left( \delta + n \right) \left( 1 - \alpha \right) t}
\]

in terms of \( k \)

\[
k = \left( \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} e^{-n \left( 1 - \alpha - \beta \right) t} + \left[ k \left( 0 \right)^{1-\alpha} - \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} \right] e^{-\left( \delta + n \right) \left( 1 - \alpha \right) t} \right)^{\frac{1}{1-\alpha}}
\]

then when \( t \to \infty \) we have that \( k \to 0 \) and \( y \equiv k^\alpha z^{1-\alpha-\beta} \to 0 \) because limits of \( k \) and \( z \) are zero. Additionally, the limit of \( K \) is calculated in the following steps

\[
\lim_{t \to \infty} kL = \left( \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} e^{-n \left( 1 - \alpha - \beta \right) t} + \left[ k \left( 0 \right)^{1-\alpha} - \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} \right] e^{-\left( \delta + n \right) \left( 1 - \alpha \right) t} \right)^{\frac{1}{1-\alpha}} L
\]

\[
= \left( \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} e^{-n \left( 1 - \alpha - \beta \right) t} + \left[ k \left( 0 \right)^{1-\alpha} - \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} \right] e^{-\left( \delta + n \right) \left( 1 - \alpha \right) t} \right)^{\frac{1}{1-\alpha}} L e^{nt}
\]

\[
= \left( \left[ k \left( 0 \right)^{1-\alpha} - \frac{\gamma}{n \beta + \delta \left( 1 - \alpha \right)} \right] e^{-\left( \delta + n \right) \left( 1 - \alpha \right) t} e^{nt} \right)^{\frac{1}{1-\alpha}} L e^{nt}
\]

thus because \( n \beta t \to \infty \) as \( t \) goes to infinity, we can say that \( \lim_{t \to \infty} \left( kL \equiv K \right) = \infty \). In the case of \( Y \) we have that \( \lim_{t \to \infty} Y = \infty \) as well because limits of \( K \) and \( L \) goes to infinity as \( t \) goes to infinity. Summarizing results we have the following

Taking conclusions from item (b) we have that returns to land diverges because \( K, L \to \infty \) as long as \( t \to \infty \):

\[
\lim_{t \to \infty} \left( P_z \equiv Y_z = \left( 1 - \alpha - \beta \right) K^\alpha L^\beta Z^{-\alpha-\beta} \right) = \infty
\]
This results is completely intuitive: the land is fixed and as long as population grows we are going to have more demand on a piece of land through time. In the case of marginal productivity of labor we have that

\[
\lim_{t \to \infty} (w \equiv Y_L = \beta K^\alpha L^{\beta-1} Z^{1-\alpha-\beta})
\]

\[
\lim_{t \to \infty} (w \equiv Y_L = \frac{\beta K^\alpha Z^{1-\alpha-\beta}}{L^{\alpha-\alpha} L^{\beta-1}})
\]

\[
\lim_{t \to \infty} (w \equiv Y_L = \beta k^\alpha z^{1-\alpha-\beta})
\]

\[
\lim_{t \to \infty} (w \equiv Y_L = k^\alpha z^{1-\alpha-\beta}) = 0
\]

Thus salary or wage goes to 0 because limits of z and k are zero.

(c) We can change a little bit the model for allowing changes in population growth rate through over time. Acemoglu’s solution book (Peter and Simsek; 2009) proposes to include a "Malthu’s argument", this is endogenize \( n \). Let’s say

<table>
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<tr>
<th>If ( y(t) )</th>
<th>Then ( n )</th>
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<tr>
<td>( y(t) &gt; y^* )</td>
<td>( n = \bar{n} &gt; 0 )</td>
</tr>
<tr>
<td>( y(t) = y^* )</td>
<td>( n = 0 )</td>
</tr>
<tr>
<td>( y(t) &lt; y^* )</td>
<td>( n = n &lt; 0 )</td>
</tr>
</tbody>
</table>

Peter and Simsek (2009) argue that result \( \lim_{t \to \infty} y = 0 \) can be avoided if we endogenize \( n \). When \( y \to 0 \) that behaviour will shrink the population growth rate \( n(y) \) which boost the growth rate of capital \( \left(\frac{c_0}{\sigma}\right) \) thus the output per capita will increase and we would expect \( \lim_{t \to \infty} y = y^* \).

2.16 Considering the solution proposal in Peter and Simsek (2009) and assuming that \( \alpha, c_0 > 0 \) and \( \alpha^{-1} + c_0 = 1 \), find the limit of production-labor ratio function \( (y) \) when \( \sigma \to 1 \).

References


about when policy reform will be more effective, and we provide preliminary evidence consistent with these ideas.6. Much of the economic literature on policy reform and most of the advice given by international institutions assume, implicitly or explicitly, that distortionary policies came about by accident. Either these policies were put in place long ago and remain as a historical legacy, or they are the outcome of some mistaken economic theory or shortsightedness on the part of policymakers. But this perspective is limited at best. Acemoglu. This volume is the Instructor Edition of the solutions manual, which contains a wider range of exercises than the Student Edition. The exercise selection for both editions is guided by a similar set of principles. First, we have tried to include the exercises that facilitate the understanding of the material covered in the book, for example, the ones that contain proofs to propositions or important extensions of the baseline models. Our solutions regularly refer to equations in the book and also to equations defined within the manual. Asymmetric Growth and Institutions in an Interdependent World Daron Acemoglu James A. Robinson Thierry Verdier Journal of Political Economy, 125(5), pp. 1245-1303. August 2017. Privacy-Constrained Network Formation Daron Acemoglu Ali Makhdoumi Azarakhsh Malekian Asuman Ozdaglar Games and Economic Behavior, 105(2017), pp. 255-275.